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Sums of Exponentials with Restricted Frequencies

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Sums of exponentials are known to have unpleasant topological and analytical properties. By restricting their frequencies they can be made better behaved.

Let n be fixed. Let $\|\cdot\|$ be the Chebyshev (sup) norm on X, a subset of at least n points of a finite interval $[\alpha, \beta]$. Consider the family of approximations of the form

$$F(A, x) = \sum_{k=1}^{n} a_k \exp(a_{n+k} x). \tag{1}$$

Associated with this family we have a long series of topological and analytical difficulties [2; 4, p. 386; 5; 8, pp. 178–179]. The classical approach to the existence difficulty is to extend the family, as in Werner [10]. We take the opposite approach: We cure the difficulties by restricting the frequencies (the a_{n+k}'s).

THEOREM. *Let $\{a^j_{n+k}\} \to a^0_{n+k}$ for $k = 1,..., n$. Let $\{a^0_{n+k} : k = 1,..., n\}$ be distinct and finite. Let $\{\|F(A^j, \cdot)\|\}$ be bounded. Then $\{A^j\}$ has a convergent subsequence $\{A^{j(i)}\}$ and the corresponding sequence $\{F(A^{j(i)}, \cdot)\}$ is uniformly convergent on $[\alpha, \beta]$ to a function of the form (1) with $a_{n+k} = a^0_{n+k}$, $k = 1,..., n$.*

Proof. Define the parameter semi-norm

$$\|A\|_c = \max\{|a_i| : i = 1,..., n\}.$$

We claim that $\{\|A^j\|_c\}$ is bounded. Suppose not, then by taking a subsequence if necessary we can assume

$$\|A^j\|_c > j. \tag{2}$$

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Consider $F(B^j, \cdot)$, where

$$b_k^j = a_k^j / \|A^j\|_c \qquad b_{n+k}^j = a_{n+k}^j,$$

then

$$|b_k^j| \leqslant 1.$$

$\{B^j\}$ has an accumulation point B^0, assume without loss of generality $\{B^j\} \to B^0$. $\{F(B^j, \cdot)\} \to F(B^0, \cdot)$ uniformly on $[\alpha, \beta]$. By (2) and boundedness of $\{\|F(A^j, \cdot)\|\}$, $\|F(B^j, \cdot)\| \to 0$ hence $\|F(B^0, \cdot)\| = 0$. But at least one of $\{b_1^0, ..., b_n^0\}$ is of magnitude 1, so this contradicts linear independence of $\{\exp(b_{n+1}x), ..., \exp(b_{2n}x))\}$ on X. Hence $\{\|A^j\|_c\}$ is bounded after all. By taking a subsequence if necessary we can assume $\{a_k^j\} \to a_k^0$, $k = 1, ..., n$ and $\{A^k\} \to A^0$, $F(A^k, \cdot) \to F(A^0, \cdot)$ uniformly on $[\alpha, \beta]$.

Since uniform convergence is conspicuously absent from subsequences of bounded sequences of exponential sums, the theorem shows that this difficulty must be caused by frequencies becoming unbounded or coalescing. This can be cured by restriction.

In [3] is given Young's condition. In it and other papers of the author [7, 8] it is shown that this property ensures nice limiting behaviour in approximation.

COROLLARY 1. $\{F(A, \cdot)$: $\mu \leqslant a_{n+k} \leqslant \nu$, a_{n+k}'s *separated by at least* $\delta\}$ *satisfies Young's condition on closed* X *with any* n *point subset being parameter bounding.*

COROLLARY 2. $\{F(A, \cdot)$: $\mu_k \leqslant a_{n+k} \leqslant \nu_k\}$, *where* $\nu_k < \mu_{k+1}$ *for* $k = 1, ..., n-1$, *satisfies Young's condition on closed* X *with any* n *point subset being parameter bounding. For* $\mu_k = \nu_k$ *we have a fixed frequency.*

It may be difficult to characterize best approximation by the families of the corollaries. But we can characterize best approximation by interior elements of such families: call $F(A, \cdot)$ an *interior* point if the inequalities are strict and separation $> \delta$ in Corollary 1 and if the inequalities are strict in Corollary 2. We claim that interior points are best with restrictions if and only if they are best from (1). Sufficiency is obvious. For necessity, suppose $f - F(A, \cdot)$ does not alternate the required number of times [9, p. 178]. Then by the proof of Meinardus and Schwedt [9, p. 144ff], there is a sequence $\{A^k\} \to A$ such that $F(A^k, \cdot)$ is better for all k sufficiently large. But A^k is in our restricted family for all k sufficiently large. If an interior point $F(A, \cdot)$ is best, it is uniquely best, for $f - F(A, \cdot)$ must alternate and we can apply the strong form of the lemma of de la Vallée-Poussin. Since the families of Corollaries 1 and 2 are subsets of (1), strong uniqueness for interior points with a_k's nonzero follows [1] with strong uniqueness constants for (1) applying.

It should be noted that if f possesses a best approximation of the form (1), it is in the families of Corollary 1 for μ sufficiently large and negative, ν sufficiently large and positive and δ sufficiently small.

Portions of the theory of the paper can be extended to approximation by other related families of approximations. In particular all of it applies to exponential-polynomial sums of [6]: the only change is that X contain at least $n+m$ points, and that frequencies be bounded away from zero.

References

1. R. Barrar and H. Loeb, On the continuity of the non-linear Tschebyscheff operator, *Pacific J. Math.* **32** (1970), 593–601.
2. C. Dunham, Approximation by alternating families on subsets, *Computing* **9** (1972), 261–265.
3. C. Dunham, The limit of non-linear Chebyshev approximation on subsets, *Aequationes Math.* **9** (1973), 60–63.
4. C. Dunham, Nearby alternating Chebyshev approximation, *Studia Sci. Math. Hungar.* **10** (1975), 381–388.
5. C. Dunham, Alternating approximation on shorter intervals, *Studia Sci. Math. Hungar* **14** (1979), 409–413.
6. C. Dunham, Chebyshev approximation by exponential-polynomial sums, *J. Comput. Appl. Math.* **5** (1979), 53–57.
7. C. Dunham, The limit of approximation with weights large on nodes, *J. Approx. Theory* **40** (1984), 155–157.
8. C. Dunham, Nearby approximation with Young's condition, *J. Approx. Theory Appl.* **1** (1984), 77–80.
9. G. Meinardus, "Approximation of Functions," Springer-Verlag, New York, 1967.
10. H. Werner, Tschebyscheff-approximation with sums of exponentials *in* "Approximation Theory," (A. Talbot, Ed.), 109–136, Academic Press, London, 1970.